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The induced representations of Brauer algebras and the Clebsch–Gordan coefficients of $SO(n)$

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Abstract. Induced representations of $D_f(n)$ from $S_{f_1} \times S_{f_2}$ with $f_1 + f_2 = f$ are discussed. The induction coefficients (IDCs) or the outer-product reduction coefficients of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f \leqslant 4$ up to a normalization factor are derived by using the linear equation method. Weyl tableaux for the corresponding Gel'fand basis of $SO(n)$ are defined. The assimilation method for obtaining Clebsch–Gordan coefficients of $SO(n)$ in the Gel'fand basis for no modification rule involved couplings from IDCs of Brauer algebras is proposed. Some isoscalar factors of $SO(n) \supset SO(n-1)$ for the resulting irrep $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dot{0}]$ with $\sum_{i=1}^{4} \lambda_i \leqslant 4$ are tabulated.

1. Introduction

Clebsch–Gordan coefficients (CGCs) are of importance in many physical problems. Besides those of $SO(3)$ and $SO(4)$, which are discussed extensively in the literature and can be found in various forms [1, 2], analytic expressions for isoscalar factors (ISFs) of $SO(n) \supset SO(n-1)$, determined using the substitution group technique [3] and which can be used to evaluate CGCs of $SO(n)$ in its canonical basis according to the Racah factorization lemma [4], are available for the coupling $[l_1, \dot{0}] \times [l_2, \dot{0}]$ to $[L_1, L_2, \dot{0}]$. In the early 1970s, the most general CGCs of $SO(n)$ for the coupling $[l_1, l_2, \ldots, l_n, \dot{0}] \times [1\dot{0}]$ and the corresponding ISFs were derived by Gavrilik [5] using the Gel'fand–Tsetlin expressions for the representation matrix of $SO(n)$ generators [6]. These CGCs in a different form were also presented in a monograph of Klimyk [7]. The complementary orthogonal and symplectic group relations [8] also allowed us to explain the analytical continuation relations between some multiplicity-free ISFs of $SO(n)$ and $SO(5)$ in $SU(2) \times SU(2)$ basis [9], which were also derived in [10]. Furthemore, ISFs for the coupling $[l_1, l_2, \dot{0}] \times [l_3, \dot{0}]$ to $[L_1, L_2, \dot{0}]$ for some special cases have been determined using the group chain transformation method [11], and a special class of multiplicity-free $O(n) \supset O(n-1)$ isoscalar factors in the Gel'fand basis were also derived in [12]. Very recently, ISFs of $O(n) \supset O(n-1)$ for the coupling $[l_1, l_2, l_3, \dot{0}] \times [1, \dot{0}]$ were derived using the irreducible tensor basis method. [13] However, in contrast to the $SU(n)$ case, CGCs of $SO(n)$ in the canonical basis or ISFs of $SO(n) \supset SO(n-1)$ are usually rank n dependent and therefore difficult to determine analytically for the general case. In most cases, analytical expressions for the CGCs are

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very complicated and difficult to integrate into applications so tables of CGCs or ISFs, if available, are useful for practical applications.

In this paper, we will outline a procedure for deriving CGCs of $SO(n)$ in its canonical basis from induction coefficients (IDCs) of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$. Brauer algebras $D_f(n)$, which are similar to the group algebra of the symmetric group S_f and which in turn are related to the decomposition of f-rank tensors of the general linear group $GL(n)$, are the centralizer algebras of the orthogonal group $O(n)$ or the symplectic group $Sp(2m)$ when $n = -2m$. More precisely, if G is the orthogonal group $O(n)$ or the symplectic group $Sp(2m)$, the corresponding centralizer algebra $B_f(G)$ are quotients of Brauer's $D_f(n)$ and $D_f(-2m)$, respectively. Hence, the duality relation between $D_f(n)$ and $O(n)$ or $Sp(2m)$ is the same as the Schur–Weyl duality relation between S_f and $GL(n)$. Irreducible representations (irreps) of $D_f(n)$ in the standard basis, i.e. the basis adapted to the chain $D_f(n) \supset D_{f-1}(n) \supset \cdots \supset D_2(n)$, have been constructed by using the induced representation and linear equation method [14], and more elaborately by Leduc and Ram using the so-called ribbon Hopf algebra approach [15]. Racah coefficients of $O(n)$ and $Sp(2m)$ were successfully derived from subduction coefficients (SDCs) of $D_f(n)$ by using the Brauer–Schur–Weyl duality relation [16]. A new simple Young diagrammatic method for Kronecker products of $O(n)$ and $Sp(2m)$ was also formulated [17] which is actually based on the induced representation theory of Brauer algebras discussed in this paper.

In section 2 a brief review of irreps of $D_f(n)$ in the standard basis will be given. Then induced representations of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f_1 + f_2 = f$ will be defined. In section 3, based on the linear equation method which has been proven to be effective in evaluating IDCs, SDCs of Hecke algebra, and SDCs of Brauer algebras, a procedure for the evaluation of IDCs of $D_f(n)$ will be outlined. In section 4, Weyl tabeaux for $SO(n)$ in its canonical basis will be defined. Then a general procedure for evaluating CGCs of $SO(n)$ in its canonical basis for couplings that require no modification rules will be outlined. Finally, in section 5, some analytical expressions for the ISFs of $SO(n) \supset SO(n-1)$ for the resulting irrep $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dot{0}]$ with $\sum_{i=1}^{4} \lambda_i \leqslant 4$ will be tabulated.

2. Brauer algebra and its outer-product basis

The Brauer algebra $D_f(n)$ is defined algebraically by $2f-2$ generators $\{g_1, g_2, \ldots, g_{f-1}, e_1$ $e_2, \ldots, e_{f-1}\}$ satisfying the following relations [18]

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad g_i g_j = g_j g_i, \ |i-j| \geqslant 2 \tag{1a}$$

$$e_i g_i = e_i \qquad e_i g_{i-1} e_i = e_i. \tag{1b}$$

Using the above defined relations and by drawing pictures of link diagrams, [18, 19] one can also derive other useful relations. For example,

$$\begin{aligned} & e_i e_j = e_j e_i \qquad |i-j| \geqslant 2 \\ & e_i^2 = n e_i \\ & (g_i - 1)^2 (g_i + 1) = 0. \end{aligned} \tag{2}$$

We assume that the base field is $\boldsymbol{C}$. The star operation, a conjugate linear map †, is defined on $D_f(n)$ by

$$g_i^\dagger = g_i \qquad e_i^\dagger = e_i \qquad \text{for } i = 1, 2, \ldots, f-1. \tag{3}$$

These are neccessary for defining orthonormal basis of $D_f(n)$.

It is easy to see that $\{g_1, g_2, \ldots, g_{f-1}\}$ generate a subalgebra $\boldsymbol{C}S_f$, which is isomorphic to the group algebra of the symmetric group; that is, $D_f(n) \supset \boldsymbol{C}S_f$. The properties of $D_f(n)$ have been discussed in [18, 19]. Based on these results, it is known that $D_f(n)$ is semisimple, i.e. it is a direct sum of a full matrix algebra over **C**, when n is not an integer or when it is an integer with $n \geqslant f-1$, otherwise $D_f(n)$ is no longer semisimple. In the following, we assume that n is an integer with $n \geqslant f-1$ and hence $D_f(n)$ is semisimple. Irreps of $D_f(n)$ can be denoted by a Young diagram with $f, f-2, f-4, \ldots, 1$ or 0 boxes. An irrep of $D_f(n)$ with $f-2k$ boxes is denoted as $[\lambda]_{f-2k}$. The branching rule of $D_f(n) \downarrow D_{f-1}(n)$ is

$$[\lambda]_{f-2k} = \oplus_{[\mu]\leftrightarrow[\lambda]}[\mu]$$

where $[\mu]$ runs through all the diagrams obtained by removing or (if $[\lambda]$ contains less than f boxes) adding a box to $[\lambda]$. Hence, the basis vectors of $D_f(n)$ in the standard basis can be denoted by

$$\left|\begin{matrix} [\lambda]_{f-2k} & D_f(n) \\ [\mu] & D_{f-1}(n) \\ \vdots & \vdots \\ [\rho] & D_{f-p+1}(n) \\ [\nu] & D_{f-p}(n) \end{matrix}\right\rangle = \left|\begin{matrix} [\lambda]_{f-2k} \\ [\mu] \\ \vdots \\ [\rho] \\ Y_M^{[\nu]} \end{matrix}\right\rangle \tag{4}$$

where $[\nu]$ is identical to the same irrep of S_{f-p}, $Y_M^{[\nu]}$ is a standard Young tableau, and M can be understood either as the Yamanouchi symbols or indices of the basis vectors in the so-called decreasing page order of the Yamanouchi labelling scheme. Irreps in the standard basis given by (4) are given in [14] for $f \leqslant 5$. Higher-dimensional results can be derived using the method outlined in [14] or by employing the Leduc and Ram formulation [15].

In order to study CGCs of $SO(n)$, we need to consider induced representations of Brauer algebra, $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f_1 + f_2 = f$, for the outer products

$$[\lambda_1] \times [\lambda_2] \uparrow \sum_{\lambda} \{\lambda_1\lambda_2\lambda\}[\lambda] \tag{5}$$

where $\{\lambda_1\lambda_2\lambda\}$ is the number of occurrences of the irrep $[\lambda]$ in the outer product $[\lambda_1] \times [\lambda_2]$. The standard basis vectors of $[\lambda_1]_{f_1}$ and $[\lambda_2]_{f_2}$ for $D_{f_1}(n)$ and $D_{f_2}(n)$, which are the same as those for S_{f_1} and S_{f_2}, can be denoted by $|Y_{m_1}^{[\lambda_1]}(\omega_1^0)\rangle$ and $|Y_{m_2}^{[\lambda_2]}(\omega_2^0)\rangle$, respectively, where

$$(\omega_1^0) = (1, 2, \ldots, f_1) \qquad (\omega_2^0) = (f_1+1, f_1+2, \ldots, f_1+f_2) \tag{6}$$

are indices in the standard tableaux $Y_{m_1}^{[\lambda_1]}$ and $Y_{m_2}^{[\lambda_2]}$, respectively. The products of the two basis vectors are denoted by

$$|Y_{m_1}^{[\lambda_1]}, Y_{m_2}^{[\lambda_2]}, (\omega_1^0), (\omega_2^0)\rangle \equiv |Y_{m_1}^{[\lambda_1]}(\omega_1^0)\rangle |Y_{m_2}^{[\lambda_2]}(\omega_2^0)\rangle \tag{7}$$

which is called a primitive uncoupled basis vector.

When n is a positive integer, tensor products of the rank-1 unit tensor operator of $O(n)$ can be used to construct the basis of $D_f(n)$ in the standard basis explicitly as done so in [14]. In this case the indices $1, 2, \ldots, f$ are used to distinguish tensor operators from different spaces. A set of corresponding indices $i_1, i_2, \ldots, i_f$ is needed to label the tensor components which can be taken as n different values, namely

$$T_{i_1}^1 T_{i_2}^2 \cdots T_{i_f}^f \equiv T_{i_1 i_2 \cdots i_f}^{12\cdots f}. \tag{8}$$

The action of g_i and e_i on (8) are given by

$$g_i T^{12\ldots i\;\;i+1\ldots f}_{j_1 j_2\ldots j_i j_{i+1}\ldots j_f} = T^{12\ldots i+1 i\ldots f}_{j_1 j_2\ldots j_i\;\;j_{i+1}\ldots j_f}$$
$$e_i T^{12\ldots i\;\;i+1\ldots f}_{j_1 j_2\ldots j_i j_{i+1}\ldots j_f} = \delta_{j_i, j_{i+1}} \sum_j {}^{(s)} T^{12\ldots i\;\;i+1\ldots f}_{j_1 j_2\ldots j\, j_{i+1}\ldots j_f} \tag{9a}$$

where the sum, $\sum^{(s)}$, on the right-hand side means†

$$\sum_j {}^{(s)} T^{12}_{jj} = \sum_{j\notin SO(2)} T^{12}_{jj} - (T^{1\;2}_{\alpha_2 -a_2} + T^{1\;\;2}_{-\alpha_2 \alpha_2}). \tag{9b}$$

In order to discuss couplings of $SO(n)$ in the canonical basis, i.e. the basis adapted to $SO(n) \supset SO(n-1) \supset SO(n-2) \supset \ldots \supset SO(2)$, the rank-1 $SO(n)$ tensor components are classified according to the $SO(n) \supset SO(n-1)$ reduction. Namely, the tensor components of j for rank-1 tensor $T^{[1]}_j$ of $SO(n)$ are labelled by $j = \pm\alpha_2, \alpha_3, \ldots, \alpha_n$, where $T^{[1]}_j$ with $j = \pm\alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ forms a rank-1 tensor of $SO(n-1)$, while $T^{[1]}_{\alpha_n}$ is a scalar of $SO(n-1)$. The minus sign introduced in (9*b*) is consistent with the Condon–Shortly phase convention [1] for CGCs of $SO(3)$. We assume that $\{T^{1\;2\ldots f}_{j_1 j_2\ldots j_f}\}$ spans an orthonormal inner-product space, namely

$$(T^{1'\;2'\ldots f'}_{j'_1 j'_2\ldots j'_f}, T^{1\;2\ldots f}_{j_1 j_2\ldots j_f}) = \prod \delta_{ii'} \delta_{j_i j'_i}. \tag{10}$$

Then, the primitive uncoupled basis vectors given by (7) can be expressed in terms of these T operators. For example, $S_1 \times S_1$ basis vector can be expressed as

$$|1\rangle = |1, 2\rangle = T^1_{i_1} T^2_{i_2}. \tag{11}$$

Other uncoupled basis vectors can be obtained by acting on (11) with g_1 and e_1.

$$|2\rangle = g_1|1\rangle = T^2_{i_1} T^1_{i_2} \qquad |3\rangle = e_1 T^1_{i_1} T^2_{i_2} = \delta_{i_1, i_2} \sum_i {}^{(s)} T^1_i T^2_i. \tag{12}$$

The left coset decomposition of $D_f(n)$ with respect to the subalgebra $S_{f_1} \times S_{f_2}$ is denoted by

$$D_f(n) = \sum_{\omega k} \oplus Q^k_\omega (S_{f_1} \times S_{f_2}) \tag{13}$$

where the left coset representatives $\{Q^k_\omega\}$ have two types of operations. One is the order-preserving permutations

$$Q^{k=0}_\omega (\omega^0_1, \omega^0_2) = (\omega_1, \omega_2) \tag{14}$$

where

$$(\omega_1) = (a_1, a_2, \ldots, a_{f_1}) \qquad (\omega_2) = (a_{f_1+1}, a_{f_1+2}, \ldots, a_f) \tag{15}$$

with $a_1 < a_2 < \cdots < a_{f_1}$, $a_{f_1+1} < a_{f_1+2} < \cdots < a_f$, and a_i represents any one of the numbers $1, 2, \ldots, f$. The other, $\{Q^{k\geqslant 1}_\omega\}$ contains k-fold trace contractions between two sets of indices (ω_1) and (ω_2). For example, in $S_2 \times S_1 \uparrow D_3(n)$ for the outer product $[2] \times [1]$, there are six elements in $\{Q^k_\omega\}$ with

$$\{Q^0_\omega\} = \{1, g_2, g_1 g_2\} \qquad \{Q^1_\omega\} = \{e_2, g_1 e_2, e_1 g_2\}. \tag{16}$$

The ordering of the sequences (ω) is specified in the following way. If there is no trace contraction, we regard the part $(\omega_1) = (a_1, a_2, \ldots, a_{f_1})$ as a vector of length f_1.

† It should be mentioned that a summation sign on the right-hand side of (3.8) in [14] should be added, which should be the same as (9*a*) given above except the phase convention. In (9*a*), the phase convention is chosen to be the same as that of Condon–Shortly for $SO(3)$ case. Therefore, the summation sign $\sum$ is now replaced by $\sum^{(s)}$.

If the last non-zero component of the vector $(\omega_1) - (\bar{\omega}_1)$ is less than zero, then we say $(\omega) \leqslant (\bar{\omega})$. This ordering of (ω_1, ω_2) is consistent with that for symmetric groups. [20] If there is a k-fold trace contraction, we regard ω^k as vector of length k with the components $(a_{i_1}a_{i'_1})(a_{i_2}a_{i'_2})\ldots(a_{i_k}a_{i'_k})$. If the last non-zero component of the vector $\omega^k - \bar{\omega}^k$ is less than zero, we say $\omega^k < \bar{\omega}^k$. The total order of $\overbrace{(\omega_1)\ (\omega_2)}^{k}$ is specified by $k = 0, 1, 2, \ldots, \min(f_1, f_2)$, where $\overbrace{(\omega_1), (\omega_2)}^{k}$ stands for k-fold contractions between indices in (ω_1) and (ω_2). For example, in $S_2 \times S_1 \uparrow D_3(n)$ for the outer product $[2] \times [1]$, the six elements are arranged as $\{1, g_2, g_1g_2, e_1g_2, g_1e_2, e_2\}$.

The uncoupled basis vectors needed in the construction of the coupled basis vectors of $[\lambda]$ for $D_f(n)$ are denoted by

$$Q^k_\omega|Y^{[\lambda_1]}_{m_1}, Y^{[\lambda_2]}_{m_2}, (\omega^0_1), (\omega^0_2)\rangle = |Y^{[\lambda_1]}_{m_1}, Y^{[\lambda_2]}_{m_2}, \overbrace{(\omega_1), (\omega_2)}^{k}\rangle. \tag{17}$$

The basis vectors of $[\lambda]_{f-2k}$ can thus be expressed in terms of the uncoupled basis vectors given by (17):

$$|[\lambda]_{f-2k}, \tau; \rho\rangle = \sum_{m_1m_2\omega\ k'} C^{[\lambda]_{f-2k}\rho;\tau}_{m_1m_2;k'\omega} Q^{k'}_\omega|Y^{[\lambda_1]}_{m_1}(\omega^0_1), Y^{[\lambda_2]}_{m_2}(\omega^0_2)\rangle \tag{18}$$

where ρ is the multiplicity label needed in the outer product $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2} \uparrow [\lambda]_{f-2k}$, τ stands for other labels needed for the irrep $[\lambda]_{f-2k}$, $0 \leqslant k' \leqslant k$, and the coefficient $C^{[\lambda]_{f-2k}\ \rho;\tau}_{m_1m_2;k'\omega}$ is $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2} \uparrow [\lambda]_{f-2k}$ IDC or the outer-product reduction coefficient (ORC).

The IDCs satisfy the following orthogonality relation:

$$\sum_{m_1m_2k'\omega m'_1m'_2k''\omega'} C^{[\lambda]_{f-2k}\ \rho;\tau}_{m_1m_2;k'\omega} C^{[\lambda']_{f-2k}\ \rho';\tau'}_{m'_1m'_2;k''\omega'} \mathcal{N}^{[\lambda_1\lambda_2]}_{m_1m_2k'\omega;m'_1m'_2k''\omega'} = \delta_{\lambda\lambda'}\delta_{\tau\tau'}\delta_{\rho\rho'} \tag{19}$$

where $\mathcal{N}^{[\lambda_1][\lambda_2]}$ is a symmetric norm matrix with elements defined by [14]

$$\mathcal{N}^{[\lambda_1][\lambda_2]}_{m_1m_2k'\omega;m'_1m'_2k''\omega'} = \langle Y^{[\lambda_1]}_{m_1}Y^{[\lambda_2]}_{m_2}; (\omega^0_1)(\omega^0_2)|Q^{k'}_\omega Q^{k''}_{\omega'}|Y^{[\lambda_1]}_{m'_1}Y^{[\lambda_2]}_{m'_2}; (\omega^0_1)(\omega^0_2)\rangle. \tag{20}$$

These matrix elements can be calculated easily using the algebraic relations of the Brauer algebra given by (1) and (2) and those given in [14]. While the coupled basis vectors $|[\lambda]_{f-2k}, \tau; \rho\rangle$ are orthonormal.

$$\langle[\lambda']_{f-2k'}; \tau', \rho'|[\lambda]_{f-2k}; \tau, \rho\rangle = \delta_{\lambda\lambda'}\delta_{\tau\tau'}\delta_{\rho\rho'}\delta_{kk'}. \tag{21}$$

3. Evaluation of the IDCs

The linear equation method (LEM) has been proved effective in deriving SDCs and IDCs of Hecke algebras [20], as well as SDCs of Brauer algebras [16]. The procedure for the evaluation of the IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ is similar to that proposed in [14].

First, applying the operators R_i ($= g_i$ or e_i) with $i = 1, 2, \ldots, f_1 + f_2 - 1$ to (18), the left-hand side of (18) becomes

$$\sum_{m_1m_2\omega k'}\sum_{\rho'\tau'} C^{[\lambda]_{f-2k}\ \rho';\tau'}_{m_1m_2;k'\omega}\langle[\lambda]_{f-2k}\tau'; \rho'|R_i|[\lambda]_{f-2k}\tau; \rho\rangle Q^{k'}_\omega|Y^{[\lambda_1]}_{m_1}Y^{[\lambda_2]}_{m_2}, (\omega^0_1)(\omega^0_2)\rangle \tag{22}$$

while the right-hand side of (18) becomes

$$\sum_{m_1m_2\omega k'} C^{[\lambda]_{f-2k}\rho;\tau}_{m_1m_2,k'\omega}(R_iQ^{k'}_\omega)|Y^{[\lambda_1]}_{m_1}Y^{[\lambda_2]}_{m_2}, (\omega^0_1)(\omega^0_2)\rangle. \tag{23}$$

Then, combining (22) and (23), we get

$$\sum_{\rho'\tau'} C^{[\lambda]_{f-2k}\rho';\tau'}_{m_1m_2,k'\omega} \langle[\lambda]_{f-2k}\tau';\rho'|R_i|[\lambda]_{f-2k}\tau;\rho\rangle = C^{[\lambda]_{f-2k}\rho'\tau'}_{m_1m_2,k'\omega'} w_i \tag{24}$$

where $C^{[\lambda]_{f-2k}\rho'\tau'}_{m_1m_2,k'\omega'} w_i$ is the coefficient in front of $Q^{k'}_{\omega}|Y^{[\lambda_1]}_{m_1}Y^{[\lambda_2]}_{m_2}, (\omega^0_1)(\omega^0_2)\rangle$ after applying R_i to the right-hand side of (18), and $\langle[\lambda]_{f-2k}\tau';\rho'|R_i|[\lambda]_{f-2k}\tau;\rho\rangle$ is matrix elements of R_i in the standard basis given in (4), which are already known [14].

The linear relations, which are part of the so-called intertwining relations among the IDCs given by (24), are sufficient to determine these IDCs up to a normalization factor [14], which, in turn, can be calculated using the orthogonality relation (19). It will be shown that the CGCs of $SO(n)$, expressed in terms of these IDCs, can be normalized according to specific cases and therefore normalization of the IDCs is not neccessary. However, the sign of the normalization factors, which gives overall phase of the IDCs should be chosen beforehand. In our calculation, the overall phase is fixed by requiring that the IDCs with $\min(\tau)$ first, then with $\min(m_1)$, and finally with smallest indices ω and k' be positive:

$$C^{[\lambda]_{f-2k}\rho;\min(\tau)}_{\min(m_1)m_2,k'=0\,\min(\omega)} \geqslant 0. \tag{25}$$

Using the algebraic relations of Brauer algebras, equation (24), and irreps of symmetric groups in the standard basis, [21] one can obtain all the IDCs of $S_{f_1}\times S_{f_2}\uparrow D_f$. In what follows, we will give a simple example of deriving the IDCs and some basic features of these coefficients.

Example 1. Deriving IDCs of $S_1\times S_1\uparrow D_2(n)$. The outer-product reduction is $[1]\times[1]\uparrow [2]+[1^2]+[0]$. In this case, equation (18) can be written as

$$|[2]\rangle = \sum_{i=1}^{3} a_i|i\rangle, |[1^2]\rangle = \sum_{i=1}^{3} b_i|i\rangle, \sum_{i=1}^{3} c_i|i\rangle \tag{26a}$$

where a_i, b_i, and c_i are the corresponding IDCs, and $|i>(i=1,2,3)$ are the uncoupled basis vectors defined by

$$|1\rangle = |1,2\rangle] \qquad |2\rangle = g_1|1\rangle \qquad |3\rangle = e_1|1\rangle. \tag{26b}$$

Applying generators g_1 and e_1, respectively, to (26a), one obtains

$$\begin{aligned} &a_1 = a_2 \qquad a_3 = -\frac{2}{n}a_1 \\ &b_1 = -b_2 \qquad b_3 = 0 \\ &c_1 = c_2 = 0 \qquad c_3 \neq 0. \end{aligned} \tag{26c}$$

The norm matrix for this case is

$$\mathcal{N}^{[1][1]} = \begin{pmatrix} 1 & \delta_{i_1i_2} & \delta_{i_1i_2} \\ \delta_{i_1i_2} & 1 & \delta_{i_1i_2} \\ \delta_{i_1i_2} & \delta_{i_1i_2} & n\delta_{i_1i_2} \end{pmatrix} \tag{26d}$$

which can be proved by using (8)–(10). Hence, the coupled basis vectors can now be written as

$$\begin{aligned} &|[2]\rangle = a_1\left(|1\rangle + |2\rangle - \frac{2}{n}|3\rangle\right) \\ &|[1^2]\rangle = b_1\,(|1\rangle - |2\rangle) \\ &|[0]\rangle = c_3|3\rangle. \end{aligned} \tag{27}$$

Using the norm matrix (26*d*), one can check that the basis vectors given by (27) are orthogonal. The normalization factors, of which the signs should be chosen according to (25), can easily be obtained by using (21) and (26)

$$a_1 = \sqrt{\frac{n}{2(n+\delta_{i_1 i_2}(n-2))}} \qquad b_1 = \sqrt{\frac{1}{2}} \qquad c_3 = \sqrt{\frac{1}{n}}. \tag{28}$$

It can easily be seen that $|3\rangle$ is a null vector when $i_1 \neq i_2$. In this case, (27) becomes outer-product basis vectors of the symmetric group $S_1 \times S_1 \uparrow S_2$. It is clear that the induced representations of $D_f(n)$ from $S_{f_1} \times S_{f_2}$ are $SO(n)$ tensor component dependent. Actually, the normalization of these basis vectors with respect to representations of $D_f(n)$ is not necessary. On the other hand, it can be easily seen from (26)–(28) that normalization factors of the IDCs are also $SO(n)$ tensor component dependent. The situation will become more complicated when $f_1 + f_2 = f \geqslant 3$. Furthermore, our purpose is to evaluate CGCs of $SO(n)$ from these IDCs. The coupled basis vectors of $SO(n)$ obtained from these IDCs through assimilation need to be normalized again. Therefore, unnormalized IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ are sufficient to be used in deriving CGCs of $SO(n)$. The method of how to evaluate $SO(n)$ CGCs from these unnormalized IDCs will be presented in the next section. Using this method, we have calculated unnormalized IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f_1 + f_2 = f \leqslant 4$. The signs of the normalization factors are all chosen to be positive, which is fixed by our phase convention (25). Therefore, only the absolute values of these normalization factors need to be determined according to different $SO(n)$ tensor components later.

For example, consider $S_2 \times S_1 \uparrow D_3(n)$ for $[1^2] \times [1] = [1^3] + [21] + [1]$. Using the above-mentioned procedure, one can easily obtain the following results

$$\begin{aligned}
|[1^3]\rangle &= \frac{1}{\sqrt{3}}(|1\rangle - |2\rangle + |3\rangle) \\
|[21]_1\rangle &= \sqrt{3}a_1\left(|2\rangle + |3\rangle - \frac{1}{n-1}(2|4\rangle + |5\rangle + |6\rangle)\right) \\
|[21]_2\rangle &= a_1\left(2|1\rangle + |2\rangle - |3\rangle + \frac{3}{n-1}(|5\rangle - |6\rangle)\right) \\
|[1][0]\rangle &= a_2|4\rangle \\
|[1][2]\rangle &= \frac{1}{\sqrt{2(n+2)(n-1)}}a_2(2|4\rangle + n(|5\rangle + |6\rangle)) \\
|[1][1^2]\rangle &= \sqrt{\frac{n}{2(n-1)}}a_2(|6\rangle - |5\rangle)
\end{aligned}$$

where $|1\rangle = |\frac{1}{2}, 3\rangle$, $|2\rangle = g_2|1\rangle$, $|3\rangle = g_1 g_2|1\rangle$, $|4\rangle = e_1 g_2|1\rangle$, $|5\rangle = g_1 e_2|1\rangle$, $|6\rangle = e_2|1\rangle$, and a_1, and a_2 are the corresponding normalization factors to be determined according to different $SO(n)$ tensor components later. In the next section, we will outline an assimilation method for evaluating CGCs of $SO(n)$ in the Gel'fand basis from these IDCs.

4. Evaluating CGCs of $SO(n)$ in its Gel'fand basis

Irreps of $SO(m)$, where $m = 2, 3, \ldots n$, can be labelled by partitions $[\lambda_{1m}\lambda_{2m}\ldots\lambda_{hm}]$ with $h = m/2$ for m even, and $h = (m-1)/2$ for m odd, which satisfy the condition

$$\begin{aligned}
&\lambda_{1m} \geqslant \lambda_{2m} \geqslant \cdots \geqslant \lambda_{hm} \geqslant 0 \qquad \text{for } m \text{ odd} \\
&\lambda_{1m} \geqslant \lambda_{2m} \geqslant \cdots \geqslant |\lambda_{hm}| \geqslant 0 \qquad \text{for } m \text{ even}
\end{aligned} \tag{29}$$

where λ_{im} $(i = 1, 2, \ldots, h)$ are all integers because we only discuss tensor representations of $SO(m)$. The partitions of two groups, for example, $SO(2p+1)$ and $SO(2p)$, in the canonical chain $SO(n) \supset \cdots \supset SO(2p+1) \supset SO(2p) \supset \cdots \supset SO(2)$ are related by the betweeness conditions

$$\lambda_{1\ 2p+1} \geqslant \lambda_{1\ 2p} \geqslant \lambda_{2\ 2p+1} \geqslant \lambda_{2\ 2p} \geqslant \cdots \geqslant \lambda_{p\ 2p+1} \geqslant |\lambda_{p\ 2p}|. \tag{30}$$

Similar to the $U(n)$ case, we can define Weyl tableau for $SO(n)$ in the Gel'fand basis:

$$W^{[\lambda]} = \begin{array}{|c|c|c|c|c|c|}\hline f_{12}\ (\pm a_2)\text{'s} & f_{13}\ a_3\text{'s} & f_{14}\ a_4\text{'s} & f_{15}\ a_5\text{'s} & f_{16}\ a_6\text{'s} & \cdots \\ \hline f_{24}\ (\pm a_4)\text{'s} & f_{25}\ a_5\text{'s} & f_{26}\ a_6\text{'s} & \cdots \\ \cline{1-4} f_{36}\ (\pm a_6)\text{'s} & \cdots\cdots \\ \cline{1-2} \cdots\cdots \\ \cline{1-1} \end{array} \tag{31}$$

where the signs in the front of a_{2k} $(k = 1, 2, \ldots)$ should always be the same. They can be taken to be all positive or all negative. The correspondence between the Weyl tableau and the Gel'fand basis is realized in the following way

$$\begin{aligned} &\pm f_{12} = \lambda_{12}, \quad f_{12} + f_{13} = \lambda_{13}, \quad f_{12} + f_{13} + f_{14} = \lambda_{14}, \ldots \\ &\pm f_{24} = \lambda_{24}, \quad f_{24} + f_{25} = \lambda_{25}, \quad f_{24} + f_{25} + f_{26} = \lambda_{26}, \ldots \\ &\pm f_{36} = \lambda_{36}, \quad f_{36} + f_{46} = \lambda_{46}, \ldots. \end{aligned} \tag{32}$$

For example, basis vectors of $SO(5) \supset SO(4) \supset SO(3) \supset SO(2)$ can be denoted either by a Gel'fand symbol or a Weyl tableau as

$$\begin{pmatrix} [\lambda_{15}\lambda_{25}] \\ [\lambda_{14}\lambda_{24}] \\ \lambda_{13} \\ \lambda_{12} \end{pmatrix} = \begin{array}{|c|c|c|c|}\hline f_{12}\ (\pm a_2)\text{'s} & f_{13}\ a_3\text{'s} & f_{14}\ a_4\text{'s} & f_{15}\ a_5\text{'s} \\ \hline f_{24}\ (\pm a_4)\text{'s} & f_{25}\ a_5\text{'s} \\ \cline{1-2} \end{array} \tag{33}$$

where

$$\begin{array}{llll} f_{12} = |\lambda_{12}| & f_{13} = \lambda_{13} - |\lambda_{12}| & f_{14} = \lambda_{14} - \lambda_{13} & f_{15} = \lambda_{15} - \lambda_{14} \\ f_{24} = |\lambda_{24}| & f_{25} = \lambda_{25} - |\lambda_{24}| & & \end{array} \tag{34}$$

and signs in the front of a_2 or a_4 should be taken as positive (negative) if $\lambda_{12} \geqslant 0$ (< 0) or $\lambda_{24} \geqslant 0$ (< 0).

An assimilation method for obtaining CGCs of $SO(n)$ from IDCs of Brauer algebra is the following. First, the one-box representation of $D_1(n)$ is just a rank-1 tensor of $SO(n)$

$$\boxed{i} \rightarrow T^{i}_{j_i} \tag{35}$$

where the index i is used to indicate that the tensor operator is in the ith space, while j_i is the tensor component, and can be taken as n different values. Then, an irrep $[\lambda]$ of $SO(n)$ can be constructed from rank-1 tensors through tensor product decomposition

$$T^1_{i_1} T^2_{i_2} \ldots T^f_{i_f} \Longrightarrow T^{[\lambda]_f}_{i_1 i_2 \ldots i_f}. \tag{36}$$

Next, the symmetry properties of f space indices $\{1, 2, \ldots, f\}$ and those tensor components $\{i_1, i_2, \ldots, i_f\}$ are the same. In other words, interchange of i and k is the same as interchange of j_i and j_k. Hence, there is a natural assimilation

$$i \longrightarrow j_i \tag{37}$$

After interchange i with j_i in basis vectors of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$, the resulting basis vectors become the corresponding $SO(n)$ orthogonal basis vectors in Weyl tableau form. This fact just reflects the Brauer–Schur–Weyl duality relation between $D_f(n)$ and $O(n)$.

For example, the basis vector of $D_1(n) \times D_1(n) \uparrow D_2(n)$ induced representation for the coupling $[1] \times [1] \uparrow [1^2]$ can be expressed as

$$\left| \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \right\rangle = \sqrt{\tfrac{1}{2}}\left(\left| \boxed{1}\,,\ \boxed{2} \right\rangle - \left| \boxed{2}\,,\ \boxed{1} \right\rangle\right). \tag{38}$$

After the assimilation, one gets the corresponding orthogonal basis vector of $SO(n) \times SO(n) \to SO(n)$ coupling with

$$\left| \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline\end{array} \right\rangle = \sqrt{\tfrac{1}{2}}\left(\left| \boxed{i_1}\,,\ \boxed{i_2} \right\rangle - \left| \boxed{i_2}\,,\ \boxed{i_1} \right\rangle\right). \tag{39}$$

The right-hand sides of (38) and (39) are the same, namely

$$\begin{aligned}\sqrt{\tfrac{1}{2}}\left(\left| \boxed{1}\,,\ \boxed{2} \right\rangle - \left| \boxed{2}\,,\ \boxed{1} \right\rangle\right) &= \sqrt{\tfrac{1}{2}}\left(\left| \boxed{i_1}\,,\ \boxed{i_2} \right\rangle - \left| \boxed{i_2}\,,\ \boxed{i_1} \right\rangle\right) \\ &= \sqrt{\tfrac{1}{2}}(1-g_1)|1,2\rangle \equiv \sqrt{\tfrac{1}{2}}\left(T^{12}_{i_1 i_2} - T^{12}_{i_2 i_1}\right).\end{aligned} \tag{40}$$

The only difference is that the space indices are interchanged with the corresponding $SO(n)$ tensor components. Furthermore, such interchange keeps the Brauer algebra action g_i or e_i on the uncoupled basis vectors unchanged. While the meaning of (38) and (39) is different, the former gives the basis vector of induced representation of $D_1(n) \times D_1(n) \uparrow D_2(n)$, the latter gives basis vector of $SO(n) \times SO(n) \to SO(n)$ in the canonical basis. This assimilation can thus be obtained just because of the Brauer–Schur–Weyl duality between $D_f(n)$ and $O(n)$. Furthermore, the phase convention of $SO(n)$ CGCs have already been determined by that of IDCs of Brauer algebra. Therefore, it is not necessary to consider the phase convention for $SO(n)$ CGCs again.

However, according to lemma 2 of [17], for the group $O(n)$, where $n = 2l$ or $2l+1$, the irrep $[\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{pn}, \dot{0}]$ is non-standard if $p > l$. In these cases, modification rules will be needed. In such circumstances, some irregular representations will be involved, which cannot be obtained by using the assimilation method. For example, coupled basis vectors of $[1] \times [1] \to [11]$ or $[1\ \ -1]$ for $SO(4)$ cannot be expressed by uncoupled basis vectors as given by (39) because the subirreps of $SO(4)$ involve the coupling $[1] \times [1] \to [11]$ for $SO(3)$. The irrep $[11] = [1]$ for $SO(3)$ obviously needs modification rules for $O(n)$. i.e. (39) is only valid for $SO(n)$ for $n \geqslant 5$. Therefore, we only consider CGCs for which no modification rule is needed in the couplings. Using this assimilation method, one can evaluate CGCs of $SO(n)$ in the canonical basis with no modification rule needed in the couplings from IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$. In the following, we will give an example to show how this method works.

Example 2. Find $SO(n)$ CGCs for $[1] \times [1] = [2] + [1^2]^* + [0]$, where $*$ indicates that this irrep is only valid for $n \geqslant 5$.

Step 1. Write the corresponding basis vectors of $S_1 \times S_1 \uparrow D_2(n)$. Using the results given in the previous section, we have

$$\left| \boxed{1}\boxed{2} \right\rangle = a_1\left(1 + g_1 - \frac{2}{n}e_1\right)\left| \boxed{1}\,,\ \boxed{2} \right\rangle \tag{41a}$$

$$\left| \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \right\rangle = \sqrt{\frac{1}{2}}(1-g_1)\left| \boxed{1}\,,\ \boxed{2} \right\rangle \tag{41b}$$

$$|[0]\rangle = \sqrt{\frac{1}{n}e_1}\left| \boxed{1}\,,\ \boxed{2} \right\rangle. \tag{41c}$$

Table 1. ISFs $\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2] \end{matrix} \middle| \begin{matrix} [\lambda] \\ [\nu] \end{matrix} \right\rangle$ of $SO(n) \supset SO(n-1)$ for $[1] \times [1] = [2] + [1^2]^{a} + [0]$.

$[\lambda]$ / $[\nu]$ \ $[\lambda_1]$ $[\lambda_2]$ / $[\nu_1]$ $[\nu_2]$	[10] [10] / [10] [10]	[10] [10] / [10] [0]	[10] [10] / [0] [10]	[10] [10] / [0] [0]
[2] / [2]	1	0	0	0
[2] / [1]	0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	0
[2] / [0]	$\sqrt{\frac{1}{n}}$	0	0	$-\sqrt{\frac{n-1}{n}}$
$[1^2]$ / [1]	0	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	0
$[1^2]$ / $[1^2]^{b}$	1	0	0	0
[0] / [0]	$\sqrt{\frac{n-1}{n}}$	0	0	$\sqrt{\frac{1}{n}}$

[a] The corresponding ISFs for this irrep are only valid for $n \geqslant 5$.
[b] It can be taken as [11] or [1-1] when $n = 5$.

Table 2. ISFs $\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2] \end{matrix} \middle| \begin{matrix} [\lambda] \\ [\nu] \end{matrix} \right\rangle$ of $SO(n) \supset SO(n-1)$ for $[2] \times [1] = [30] + [21]^{a} + [1]$.

$[\lambda]$ / $[\nu]$ \ $[\lambda_1]$ $[\lambda_2]$ / $[\nu_1]$ $[\nu_2]$	[2] [1] / [2] [1]	[2] [1] / [2] [0]	[2] [1] / [1] [1]	[2] [1] / [1] [0]	[2] [1] / [0] [1]	[2] [1] / [0] [0]
[3] / [3]	1	0	0	0	0	0
[3] / [2]	0	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	0	0	0
[3] / [1]	$\sqrt{\frac{2(n-2)}{3(n+2)(n-1)}}$	0	0	$\sqrt{\frac{2(n+1)}{3(n+2)}}$	$-\sqrt{\frac{n(n+1)}{3(n+2)(n-1)}}$	0
[3] / [0]	0	0	$\sqrt{\frac{2}{n+2}}$	0	0	$\sqrt{\frac{n}{n+2}}$
[21] / $[21]^{b}$	1	0	0	0	0	0
[21] / [2]	0	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$	0	0	0
[21] / [1]	$\sqrt{\frac{n+1}{3(n-1)^2}}$	0	0	$\sqrt{\frac{n-2}{3(n-1)}}$	$\sqrt{\frac{2n(n-2)}{3(n-1)^2}}$	0
[21] / $[1^2]^{c}$	0	0	1	0	0	0
[1] / [1]	$\sqrt{\frac{n(n+1)(n-2)}{(n+2)(n-1)^2}}$	0	0	$-\sqrt{\frac{n}{(n+2)(n-1)}}$	$-\sqrt{\frac{2}{(n+2)(n-1)^2}}$	0
[1] / [0]	0	0	$\sqrt{\frac{n}{n+2}}$	0	0	$-\sqrt{\frac{2}{n+2}}$

[a] The corresponding ISFs for this irrep are only valid for $n \geqslant 5$.
[b] It can be taken as [21] or [2-1] when $n = 5$.
[c] It can be taken as [11] or [1-1] when $n = 5$.

Step 2. Make assimilations. We need to consider three different ways of assimilation in this case.

Table 3. ISFs $\left\langle \begin{matrix}[\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2]\end{matrix} \middle| \begin{matrix}[\lambda] \\ [\nu]\end{matrix} \right\rangle$ of $SO(n) \supset SO(n-1)$ for $[30] \times [10] = [40] + [31]^{a} + [20]$.

$[\lambda]$ / $[\nu]$	[3] [1] / [3] [1]	[3] [1] / [3] [0]	[3] [1] / [2] [1]	[3] [1] / [2] [0]	[3] [1] / [1] [1]	[3] [1] / [1] [0]	[3] [1] / [0] [1]	[3] [1] / [0] [0]
[4] [4]	1	0	0	0	0	0	0	0
[4] [3]	0	$\frac{1}{2}$	$\sqrt{\frac{3}{4}}$	0	0	0	0	0
[4] [2]	$\sqrt{\frac{n-1}{2(n+1)(n+4)}}$	0	0	$-\sqrt{\frac{n+3}{2(n+4)}}$	$\sqrt{\frac{(n+2)(n+3)}{2(n+1)(n+4)}}$	0	0	0
[4] [1]	0	0	$\sqrt{\frac{3(n-2)}{2(n-1)(n+4)}}$	0	0	$\sqrt{\frac{3(n+2)}{4(n+4)}}$	$\sqrt{\frac{(n+1)(n+2)}{4(n-1)(n+4)}}$	0
[4] [0]	0	0	0	0	$\sqrt{\frac{3}{n+4}}$	0	0	$\sqrt{\frac{n+1}{n+4}}$
[31] [3]	0	$\sqrt{\frac{3}{4}}$	$-\frac{1}{2}$	0	0	0	0	0
[31] [2]	$\sqrt{\frac{n+3}{2n(n+1)}}$	0	0	$-\sqrt{\frac{n-1}{2n}}$	$-\sqrt{\frac{(n-1)(n+2)}{2n(n+1)}}$	0	0	0
[31] [1]	0	0	$\sqrt{\frac{n+2}{2n(n-1)}}$	0	0	$\sqrt{\frac{n-2}{4n}}$	$-\sqrt{\frac{3(n+1)(n-2)}{4n(n-1)}}$	0
[31] [21][b]	0	0	1	0	0	0	0	0
[31] $[1^2]$[c]	0	0	0	0	1	0	0	0
[2] [2]	$\sqrt{\frac{(n+2)(n+3)(n-1)}{n(n+1)(n+4)}}$	0	0	$\sqrt{\frac{n+2}{n(n+4)}}$	$\sqrt{\frac{4}{n(n+1)(n+4)}}$	0	0	0
[2] [1]	0	0	$\sqrt{\frac{(n^2-4)(n+1)}{n(n-1)(n+4)}}$	0	0	$-\sqrt{\frac{2(n+1)}{n(n+4)}}$	$\sqrt{\frac{6}{n(n-1)(n+4)}}$	0
[2] [0]	0	0	0	0	$\sqrt{\frac{n+1}{n+4}}$	0	0	$-\sqrt{\frac{3}{n+4}}$

[a] The corresponding ISFs for this irrep are only valid for $n \geqslant 5$.
[b] It can be taken as [21] or [2-1] when $n = 5$.
[c] It can be taken as [11] or [1-1] when $n = 5$.

(a) $i_1 = \tau\alpha_k$, $i_2 = \delta\alpha_m$ $(k < m)$, where τ can be taken as $-$ for $k = 2$, and $\tau = +$ for other cases, and δ can be taken as $-$ when α_m is in the second row for $m = 4$, and can only be taken as $+$ for other cases. Because the tensor indices are different, contraction of i_1 and i_2 is zero. Hence, we get

$$\left| \boxed{\tau\alpha_k}\boxed{\alpha_m} \right\rangle = \sqrt{\tfrac{1}{2}}\left(\left| \boxed{\tau\alpha_k}, \boxed{\alpha_m} \right\rangle + \left| \boxed{\alpha_m}, \boxed{\tau\alpha_k} \right\rangle \right) \qquad (42a)$$

$$\left| \begin{array}{|c|}\hline \tau\alpha_k \\ \hline \delta\alpha_m \\ \hline\end{array} \right\rangle = \sqrt{\tfrac{1}{2}}\left(\left| \boxed{\tau\alpha_k}, \boxed{\alpha_m} \right\rangle - \left| \boxed{\alpha_m}, \boxed{\tau\alpha_k} \right\rangle \right) \qquad (42b)$$

where the normalization factor $a_1 = \sqrt{\frac{1}{2}}$. It should be noted that the $SO(2m)$ tensor indices $\delta\alpha_{2m}$ can be taken $-\alpha_{2m}$ only in the mth row in the Weyl tableau. $-\alpha_{2m}$ in the pth row with $p < m$ is forbidden according to the definition of $SO(n)$ Weyl tableau. Hence, $\delta\alpha_m$ should be replaced by α_m in the pth row with $p < m$. Thus, δ can be taken as $-$ only for $m = 4$, and the only possible minus sign of τ allowed in the first row is $k = 2$ case. This result is mainly due to the $O(2n) \downarrow SO(2n)$ reduction $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n\} \downarrow [\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n]$ $+[\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, -\lambda_n]$ if $\lambda_n \neq 0$. For example, from (42a), for $m = 3$ and $k = 2$, one

Table 4. ISFs $\left\langle\begin{matrix}[\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2]\end{matrix}\right|\left.\begin{matrix}[\lambda] \\ [\nu]\end{matrix}\right\rangle$ of $SO(n) \supset SO(n-1)$ for $[1^2] \times [1] = [21] + [1^3] + [1]$ for $n \geqslant 7$.

$[\lambda]$ $[\nu]$ / $[\lambda_1]$ $[\lambda_2]$ $[\nu_1]$ $[\nu_2]$	$[1^2]$ $[10]$ $[1^2]$ $[10]$	$[1^2]$ $[10]$ $[1^2]$ $[0]$	$[1^2]$ $[10]$ $[1]$ $[1]$	$[1^2]$ $[10]$ $[1]$ $[0]$
[21] [21]	1	0	0	0
[21] [2]	0	0	1	0
[21] $[1^2]$	0	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	0
[21] [1]	$\sqrt{\frac{1}{n-1}}$	0	0	$\sqrt{\frac{n-2}{n-1}}$
$[1^3]$ $[1^3]$[a]	1	0	0	0
$[1^3]$ [1]	0	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	0
[1] [1]	$\sqrt{\frac{n-2}{n-1}}$	0	0	$-\sqrt{\frac{1}{n-1}}$
[1] [0]	0	0	1	0

[a] It can be taken as [111] or [11-1] when $n = 7$.

Table 5. ISFs $\left\langle\begin{matrix}[\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2]\end{matrix}\right|\left.\begin{matrix}[\lambda] \\ [\nu]\end{matrix}\right\rangle$ of $SO(n) \supset SO(n-1)$ for $[1^3] \times [1] = [211] + [1^4] + [1^2]$ for $n \geqslant 9$.

$[\lambda]$ $[\nu]$ / $[\lambda_1]$ $[\lambda_2]$ $[\nu_1]$ $[\nu_2]$	$[1^3]$ $[10]$ $[1^3]$ $[10]$	$[1^3]$ $[10]$ $[1^3]$ $[0]$	$[1^3]$ $[10]$ $[1^2]$ $[1]$	$[1^3]$ $[10]$ $[1^2]$ $[0]$
[211] [211]	1	0	0	0
[211] [21]	0	0	1	0
[211] $[1^2]$	$\sqrt{\frac{1}{n-2}}$	0	0	$\sqrt{\frac{n-3}{n-2}}$
[211] $[1^3]$	0	$\sqrt{\frac{3}{4}}$	$\frac{1}{2}$	0
$[1^4]$ $[1^4]$[a]	1	0	0	0
$[1^4]$ $[1^3]$	0	$\frac{1}{2}$	$-\sqrt{\frac{3}{4}}$	0
$[1^2]$ $[1^2]$	$\sqrt{\frac{n-3}{n-2}}$	0	0	$-\sqrt{\frac{1}{n-2}}$
$[1^2]$ [1]	0	0	1	0

[a] It can be taken as [1111] or [111-1] when $n = 9$.

gets the following CGCs of $SO(3)$

$$\left\langle\begin{matrix}[1] & [1] \\ \pm1 & 0\end{matrix}\right|\left.\begin{matrix}[2] \\ \pm1\end{matrix}\right\rangle = \sqrt{\frac{1}{2}} \qquad \left\langle\begin{matrix}[1] & [1] \\ 0 & \pm1\end{matrix}\right|\left.\begin{matrix}[2] \\ \pm1\end{matrix}\right\rangle = \sqrt{\frac{1}{2}}. \tag{43a}$$

Table 6. ISFs $\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2] \end{matrix} \middle| \begin{matrix} [\lambda] \\ [\nu] \end{matrix} \right\rangle$ of $SO(n) \supset SO(n-1)$ for $[2]\times[1^2] = [31]+[211]+[2]+[1^2]$ for $n \geqslant 7$.

$[\lambda]$ / $[\lambda_1]$ $[\lambda_2]$ $[\nu]$ / $[\nu_1]$ $[\nu_2]$	$[2]$ $[1^2]$ $[2]$ $[1^2]$	$[2]$ $[1^2]$ $[2]$ $[1]$	$[2]$ $[1^2]$ $[1]$ $[1^2]$	$[2]$ $[1^2]$ $[1]$ $[1]$	$[2]$ $[1^2]$ $[0]$ $[1^2]$	$[2]$ $[1^2]$ $[0]$ $[1]$
$[31]$ $[31]$	1	0	0	0	0	0
$[31]$ $[3]$	0	1	0	0	0	0
$[31]$ $[21]$	0	$-\frac{1}{2}$	$\sqrt{\frac{3}{4}}$	0	0	0
$[31]$ $[2]$	$\sqrt{\frac{1}{n}}$	0	0	$-\sqrt{\frac{n-1}{n}}$	0	0
$[31]$ $[1^2]$	$\sqrt{\frac{n-3}{2(n+2)(n-1)}}$	0	0	$\sqrt{\frac{n+1}{2(n+2)}}$	$-\sqrt{\frac{n(n+1)}{2(n+2)(n-1)}}$	0
$[31]$ $[1]$	0	$\sqrt{\frac{2}{n(n+2)(n-1)}}$	$\sqrt{\frac{2(n+1)}{n(n+2)}}$	0	0	$\sqrt{\frac{(n+1)(n-2)}{(n-1)(n+2)}}$
$[211]$ $[211]$[a]	1	0	0	0	0	0
$[211]$ $[21]$	0	$\sqrt{\frac{3}{4}}$	$\frac{1}{2}$	0	0	0
$[211]$ $[1^3]$[b]	0	0	1	0	0	0
$[211]$ $[1^2]$	$\sqrt{\frac{n+1}{2(n-2)(n-1)}}$	0	0	$\sqrt{\frac{n-3}{2(n-2)}}$	$-\sqrt{\frac{n(n-3)}{2(n-2)(n-1)}}$	0
$[2]$ $[2]$	$\sqrt{\frac{n-1}{n}}$	0	0	$\sqrt{\frac{1}{n}}$	0	0
$[2]$ $[1]$	0	$\sqrt{\frac{(n-2)(n+1)}{2n(n-1)}}$	$\sqrt{\frac{n-2}{2n}}$	0	0	$-\sqrt{\frac{1}{n-1}}$
$[2]$ $[0]$	0	0	0	1	0	0
$[1^2]$ $[1^2]$	$\sqrt{\frac{n(n+1)(n-3)}{(n-1)(n^2-4)}}$	0	0	$-\sqrt{\frac{n}{(n+2)(n-2)}}$	$\sqrt{\frac{4}{(n^2-4)(n-1)}}$	0
$[1^2]$ $[1]$	0	$\sqrt{\frac{n(n+1)}{2(n+2)(n-1)}}$	$-\sqrt{\frac{n}{2(n+2)}}$	0	0	$\sqrt{\frac{n-2}{(n+2)(n-1)}}$

[a] It can be taken as [211] or [21-1] when $n = 7$.
[b] It can be taken as [111] or [11-1] when $n = 7$.

From (42*b*) for $k = 3$ and $m = 5$, one gets the following CGCs for $SO(5)$

$$\left(\begin{matrix} [1] & [1] \\ [1] & 0 \\ [1] & 0 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} [11] \\ [1] \\ [1] \\ 0 \end{matrix}\right) = \sqrt{\tfrac{1}{2}} \qquad \left(\begin{matrix} [1] & [1] \\ [0] & [1] \\ [0] & [1] \\ 0 & 0 \end{matrix} \middle| \begin{matrix} [11] \\ [1] \\ [1] \\ 0 \end{matrix}\right) = -\sqrt{\tfrac{1}{2}}. \tag{43b}$$

For $m \leqslant 4$ in (42*b*) there will be representations involving modification rules, which will not be considered in this paper.

(*b*) $i_1 = i_2 = a_n$. In this case, the trace contraction is non-zero. We have

$$e_1 \left| \; \boxed{\alpha_n} \; , \; \boxed{\alpha_n} \; \right\rangle = \sum_{\mu}{}^{(s)} \left| \; \boxed{\alpha_\mu} \; , \; \boxed{\alpha_\mu} \; \right\rangle \tag{44a}$$

Table 7. ISFs $\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2] \end{matrix} \middle| \begin{matrix} [\lambda] \\ [\nu] \end{matrix} \right\rangle$ of $SO(n) \supset SO(n-1)$ for $[1^2] \times [1^2] = [22] + [21^2] + [1^4]^{a} + [20] + [1^2] + [0]$ for $n \geqslant 7$.

$[\lambda]$ / $[\nu]$; $[\lambda_1]\ [\lambda_2]$ / $[\nu_1]\ [\nu_2]$	$[1^2]\ [1^2]$ / $[1^2]\ [1^2]$	$[1^2]\ [1^2]$ / $[1^2]\ [1]$	$[1^2]\ [1^2]$ / $[1]\ [1^2]$	$[1^2]\ [1^2]$ / $[1]\ [1]$
[22] / [22]	1	0	0	0
[22] / [21]	0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	0
[22] / [2]	$\sqrt{\frac{1}{n-2}}$	0	0	$\sqrt{\frac{n-3}{n-2}}$
[211] / [211][b]	1	0	0	0
$[21^2]$ / [21]	0	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	0
$[21^2]$ / $[1^2]$	0	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	0
$[21^2]$ / $[1^2]$	$\sqrt{\frac{1}{n-2}}$	0	0	$\sqrt{\frac{n-3}{n-2}}$
$[1^4]$ / $[1^4]$[c]	1	0	0	0
$[1^4]$ / $[1^3]$	0	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	0
[2] / [2]	$\sqrt{\frac{n-3}{n-2}}$	0	0	$-\sqrt{\frac{1}{n-2}}$
[2] / [1]	0	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	0
[2] / [0]	$\sqrt{\frac{2}{n}}$	0	0	$\sqrt{\frac{n-2}{n}}$
$[1^2]$ / $[1^2]$	$\sqrt{\frac{n-3}{n-2}}$	0	0	$-\sqrt{\frac{1}{n-2}}$
$[1^2]$ / [1]	0	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	0
[0] / [0]	$\sqrt{\frac{n-2}{n}}$	0	0	$-\sqrt{\frac{2}{n}}$

[a] The corresponding ISFs for this irrep are only valid for $n \geqslant 9$.
[b] It can be taken as [211] or [21-1] when $n = 7$.
[c] It can be taken as [1111] or [111-1] when $n = 9$.

where there are n terms involved in the sum. It should be understood that the sum on the right-hand side of (44*a*) is shorthand notation, of which the exact expression should be

$$\sum_{\mu}{}^{(s)} \left| \boxed{\alpha_\mu}\ ,\ \boxed{\alpha_\mu} \right\rangle = \sum_{\mu \geqslant 3} \left| \boxed{\alpha_\mu}\ ,\ \boxed{\alpha_\mu} \right\rangle + \sum_{\delta=+,-} (-) \left| \boxed{\delta\alpha_2}\ ,\ \boxed{-\delta\alpha_2} \right\rangle . \tag{44b}$$

Thus, we get

$$\left| \boxed{\alpha_n}\boxed{\alpha_n} \right\rangle = 2a_1 \left(\left(1 - \frac{1}{n}\right) \left| \boxed{\alpha_n}\ ,\ \boxed{\alpha_n} \right\rangle - \frac{1}{n} \sum_{\mu \neq n}{}^{(s)} \left| \boxed{\alpha_\mu}\ ,\ \boxed{\alpha_\mu} \right\rangle \right) . \tag{45}$$

Table 8. ISFs $\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2] \end{matrix} \middle| \begin{matrix} [\lambda] \\ [\nu] \end{matrix} \right\rangle$ of $SO(n) \supset SO(n-1)$ for $[21] \times [10] = [31] + [22] + [211] + [20] + [1^2]$ for $n \geqslant 7$.

$[\lambda]$ / $[\nu]$	$[21]\ [1]$ / $[21]\ [1]$	$[21]\ [1]$ / $[21]\ [0]$	$[21]\ [1]$ / $[2]\ [1]$	$[21]\ [1]$ / $[2]\ [0]$	$[21]\ [1]$ / $[1^2]\ [1]$	$[21]\ [1]$ / $[1^2]\ [0]$	$[21]\ [1]$ / $[1]\ [1]$	$[21]\ [1]$ / $[1]\ [0]$
[31] / [31]	1	0	0	0	0	0	0	0
[22] / [22]	1	0	0	0	0	0	0	0
[31] / [3]	0	0	1	0	0	0	0	0
[31] / [21]	0	$\sqrt{\frac{3}{8}}$	$\frac{1}{4}$	0	$\frac{3}{4}$	0	0	0
[31] / [2]	$\sqrt{\frac{n-3}{2n(n-2)}}$	0	0	$\sqrt{\frac{n-1}{2n}}$	$\sqrt{\frac{(n-1)^2}{2n(n-2)}}$	0	0	0
[31] / $[1^2]$	$\sqrt{\frac{3(n-3)}{4(n^2-4)}}$	0	0	0	0	$-\sqrt{\frac{3(n+1)}{4(n+2)}}$	$-\sqrt{\frac{n^2-1}{4(n^2-4)}}$	0
[31] / [1]	0	0	$\sqrt{\frac{n-1}{2n(n+2)}}$	0	$-\sqrt{\frac{3(n+1)}{2n(n+2)}}$	0	0	$-\sqrt{\frac{n^2-1}{n(n+2)}}$
[22] / [21]	0	$\frac{1}{2}$	$\sqrt{\frac{3}{8}}$	0	$-\sqrt{\frac{3}{8}}$	0	0	0
[22] / [2]	$\sqrt{\frac{n-1}{2(n-2)^2}}$	0	0	$\sqrt{\frac{n-3}{2(n-2)}}$	0	0	$-\sqrt{\frac{(n-1)(n-3)}{2(n-2)^2}}$	0
$[21^2]$ / $[21^2]$[a]	1	0	0	0	0	0	0	0
$[21^2]$ / [21]	0	$\sqrt{\frac{3}{8}}$	$-\frac{3}{4}$	0	$\frac{-1}{4}$	0	0	0
[211] / $[1^3]$[b]	0	0	0	0	1	0	0	0
[211] / $[1^2]$	$\sqrt{\frac{n+1}{4(n-2)^2}}$	0	0	0	0	$-\sqrt{\frac{n-3}{4(n-2)}}$	$\sqrt{\frac{3(n-1)(n-3)}{4(n-2)^2}}$	0
[2] / [2]	$\sqrt{\frac{(n-1)^2(n-3)}{n(n-2)^2}}$	0	0	$-\sqrt{\frac{n-1}{n(n-2)}}$	0	0	$\sqrt{\frac{1}{n(n-2)^2}}$	0
[2] / [1]	0	0	$\sqrt{\frac{n+1}{4n}}$	0	$\sqrt{\frac{3(n-1)}{4n}}$	0	0	$-\sqrt{\frac{1}{2n}}$
[2] / [0]	0	0	0	0	0	0	1	0
$[1^2]$ / $[1^2]$	$\sqrt{\frac{(n^2-1)(n-3)}{(n+2)(n-2)^2}}$	0	0	0	0	$\sqrt{\frac{n-1}{n^2-4}}$	$-\sqrt{\frac{3}{(n+2)(n-2)^2}}$	0
$[1^2]$ / [1]	0	0	$\sqrt{\frac{3(n+1)}{4(n+2)}}$	0	$-\sqrt{\frac{n-1}{4(n+2)}}$	0	0	$\sqrt{\frac{3}{2(n+2)}}$

[a] It can be taken as [211] or [21-1] when $n = 7$.
[b] It can be taken as [111] or [11-1] when $n = 7$.

After normalization, (45) becomes

$$\left| \boxed{\alpha_n}\boxed{\alpha_n} \right\rangle = \sqrt{\frac{n-1}{n}} \left| \boxed{\alpha_n} , \boxed{\alpha_n} \right\rangle - \sqrt{\frac{1}{n(n-1)}} \sum_{\mu \neq n}^{(s)} \left| \boxed{\alpha_\mu} , \boxed{\alpha_\mu} \right\rangle . \quad (46)$$

Similarly, we have

$$|[0]\rangle = \sqrt{\frac{1}{n}} \left| \boxed{\alpha_n} , \boxed{\alpha_n} \right\rangle + \sqrt{\frac{1}{n}} \sum_{\mu \neq n}^{(s)} \left| \boxed{\alpha_\mu} , \boxed{\alpha_\mu} \right\rangle . \quad (47)$$

Table 9. ISFs $\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2] \end{matrix} \middle| \begin{matrix} [\lambda] \\ [\nu] \end{matrix} \right\rangle$ of $SO(n) \supset SO(n-1)$ for $[20] \times [20] = [40] + [31] + [22] + [20] + [1^2] + [0]$ for $n \geqslant 5$.

[λ] / [ν]	[2] [2] / [2] [2]	[2] [2] / [2] [1]	[2] [2] / [2] [0]	[2] [2] / [1] [2]	[2] [2] / [1] [1]	[2] [2] / [1] [0]	[2] [2] / [0] [2]	[2] [2] / [0] [1]	[2] [2] / [0] [0]
[4] [4]	1	0	0	0	0	0	0	0	0
[4] [3]	0	$\sqrt{\frac{1}{2}}$	0	$\sqrt{\frac{1}{2}}$	0	0	0	0	0
[4] [2]	$\sqrt{\frac{2(n-3)}{3(n+4)(n-1)}}$	0	$-\sqrt{\frac{n(n+3)}{6(n+4)(n-1)}}$	0	$\sqrt{\frac{2(n+3)}{3(n+4)}}$	0	$-\sqrt{\frac{n(n+3)}{6(n+4)(n-1)}}$	0	0
[4] [1]	0	$\sqrt{\frac{n-2}{(n+4)(n-1)}}$	0	$-\sqrt{\frac{n-2}{(n+4)(n-1)}}$	0	$-\sqrt{\frac{n(n+1)}{2(n-1)(n+4)}}$	0	$-\sqrt{\frac{n(n+1)}{2(n-1)(n+4)}}$	0
[4] [0]	$\sqrt{\frac{2(n-2)}{(n+4)(n-1)(n+2)}}$	0	0	0	$\sqrt{\frac{4(n+1)}{(n+2)(n+4)}}$	0	0	0	$\sqrt{\frac{n^2(n+1)}{(n+4)(n+2)(n-1)}}$
[31] [31][a]	1	0	0	0	0	0	0	0	0
[31] [3]	0	$\sqrt{\frac{1}{2}}$	0	$-\sqrt{\frac{1}{2}}$	0	0	0	0	0
[31] [2]	0	0	$\sqrt{\frac{1}{2}}$	0	0	0	$-\sqrt{\frac{1}{2}}$	0	0
[31] [1]	0	$\sqrt{\frac{n}{(n+2)(n-1)}}$	0	$\sqrt{\frac{n}{(n+2)(n-1)}}$	0	$-\sqrt{\frac{(n-2)(n+1)}{2(n-1)(n+2)}}$	0	$\sqrt{\frac{(n-2)(n+1)}{2(n-1)(n+2)}}$	0
[31] [21][b]	0	$\sqrt{\frac{1}{2}}$	0	$-\sqrt{\frac{1}{2}}$	0	0	0	0	0
[31] $[1^2]$[c]	$\sqrt{\frac{1}{n+2}}$	0	0	0	$\sqrt{\frac{n+1}{n+2}}$	0	0	0	0
[22] [22][d]	1	0	0	0	0	0	0	0	0

Table 9. (Continued)

[λ] / [ν]	[2] [2] / [2] [2]	[2] [2] / [2] [1]	[2] [2] / [2] [0]	[2] [2] / [1] [2]	[2] [2] / [1] [1]	[2] [2] / [1] [0]	[2] [2] / [0] [2]	[2] [2] / [0] [1]	[2] [2] / [0] [0]
[22] [21][b]	0	$\sqrt{\frac{1}{2}}$	0	$\sqrt{\frac{1}{2}}$	0	0	0	0	0
[22] [2]	$\sqrt{\frac{n+3}{3(n-2)(n-1)}}$	0	$\sqrt{\frac{n(n-3)}{3(n-2)(n-1)}}$	0	$\sqrt{\frac{n-3}{3(n-2)}}$	0	$\sqrt{\frac{n(n-3)}{3(n-2)(n-1)}}$	0	0
[2] [2]	$\sqrt{\frac{n(n^2-9)}{(n+4)(n-2)(n-1)}}$	0	$-\sqrt{\frac{4}{(n+4)(n-2)(n-1)}}$	0	$-\sqrt{\frac{n}{(n+4)(n-2)}}$	0	$-\sqrt{\frac{4}{(n+4)(n-2)(n-1)}}$	0	0
[2] [1]	0	$\sqrt{\frac{n(n+1)}{2(n+4)(n-1)}}$	0	$-\sqrt{\frac{n(n+1)}{2(n+4)(n-1)}}$	0	$\sqrt{\frac{n-2}{(n-1)(n+4)}}$	0	$\sqrt{\frac{n-2}{(n-1)(n+4)}}$	0
[2] [0]	$\sqrt{\frac{2(n+1)}{(n+4)(n-1)}}$	0	0	0	$\sqrt{\frac{n-2}{n+4}}$	0	0	0	$-\sqrt{\frac{4(n-2)}{(n+4)(n-1)}}$
$[1^2]$ $[1^2]$[c]	$\sqrt{\frac{n+1}{n+2}}$	0	0	0	$-\sqrt{\frac{1}{n+2}}$	0	0	0	0
$[1^2]$ [1]	0	$\sqrt{\frac{(n+1)(n-2)}{2(n+2)(n-1)}}$	0	$\sqrt{\frac{(n+1)(n-2)}{2(n+2)(n-1)}}$	0	$\sqrt{\frac{n}{(n-1)(n+2)}}$	0	$-\sqrt{\frac{n}{(n-1)(n+2)}}$	0
[0] [0]	$\sqrt{\frac{(n+1)(n-2)}{(n-1)(n+2)}}$	0	0	0	$-\sqrt{\frac{2}{n+2}}$	0	0	0	$\sqrt{\frac{2}{(n+2)(n-1)}}$

[a] It can be taken as [31] or [3-1] when $n = 5$.
[b] It can be taken as [21] or [2-1] when $n = 5$.
[c] It can be taken as [11] or [1-1] when $n = 5$.
[d] It can be taken as [22] or [2-2] when $n = 5$.

(c) $i_1 = i_2 = a_k$ $(2 < k < n)$. The final results in this case are similar to those of (46) and (47):

$$\left| \,\boxed{\alpha_k \,|\, \alpha_k}\, \right\rangle = \sqrt{\frac{k-1}{k}} \left| \,\boxed{\alpha_k}\,,\ \boxed{\alpha_k}\, \right\rangle - \sqrt{\frac{1}{k(k-1)}} \sum_{\mu\neq k}{}^{(s)} \left| \,\boxed{\alpha_\mu}\,,\ \boxed{\alpha_\mu}\, \right\rangle. \tag{48}$$

$$|[0]\rangle = \sqrt{\frac{1}{k}} \left| \,\boxed{\alpha_k}\,,\ \boxed{\alpha_k}\, \right\rangle + \sqrt{\frac{1}{k}} \sum_{\mu\neq k}{}^{(s)} \left| \,\boxed{\alpha_\mu}\,,\ \boxed{\alpha_\mu}\, \right\rangle. \tag{49}$$

The corresponding CGCs of $SO(n)$ can now be read off from ($42a, b$), (46)–(48). When $n = 4$, for example, the $SO(4) \supset SO(3) \supset SO(2)$ CGCs read off from (46) and (47) are

$$\begin{gathered}
\left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [0] & [0] & [0] \\ 0 & 0 & 0 \end{array} \right\rangle = \sqrt{\tfrac{3}{4}} \qquad
\left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [1] & [1] & [0] \\ 1 & -1 & 0 \end{array} \right\rangle = \left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [1] & [1] & [0] \\ -1 & 1 & 0 \end{array} \right\rangle = \sqrt{\tfrac{1}{12}} \\
\left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [1] & [1] & [0] \\ 0 & 0 & 0 \end{array} \right\rangle = -\sqrt{\tfrac{1}{12}} \qquad
\left\langle \begin{array}{cc|c} [1] & [1] & [0] \\ [0] & [0] & [0] \\ 0 & 0 & 0 \end{array} \right\rangle = \left\langle \begin{array}{cc|c} [1] & [1] & [0] \\ [1] & [1] & [0] \\ 0 & 0 & 0 \end{array} \right\rangle = \sqrt{\tfrac{1}{4}} \\
\left\langle \begin{array}{cc|c} [1] & [1] & [0] \\ [1] & [1] & [0] \\ 1 & -1 & 0 \end{array} \right\rangle = \left\langle \begin{array}{cc|c} [1] & [1] & [0] \\ [1] & [1] & [0] \\ -1 & 1 & 0 \end{array} \right\rangle = -\sqrt{\tfrac{1}{4}}.
\end{gathered} \tag{50}$$

When $n = 4$ and $k = 3$, from (48) and (49), we have

$$\left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [1] & [1] & [2] \\ 0 & 0 & 0 \end{array} \right\rangle = \sqrt{\tfrac{2}{3}} \qquad
\left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [1] & [1] & [2] \\ 1 & -1 & 0 \end{array} \right\rangle = \left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [1] & [1] & [2] \\ -1 & 1 & 0 \end{array} \right\rangle = \sqrt{\tfrac{1}{6}} \tag{51}$$

$$\left\langle \begin{array}{cc|c} [1] & [1] & [0] \\ 0 & 0 & 0 \end{array} \right\rangle = \sqrt{\tfrac{1}{3}} \qquad
\left\langle \begin{array}{cc|c} [1] & [1] & [0] \\ 1 & -1 & 0 \end{array} \right\rangle = \left\langle \begin{array}{cc|c} [1] & [1] & [0] \\ -1 & 1 & 0 \end{array} \right\rangle = -\sqrt{\tfrac{1}{3}} \tag{52}$$

where (52) gives CGCs of $SO(3)$.

Using this method, we have derived modification-rule-free CGCs of $SO(n)$ for the resulting irrep $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dot{0}]$ with $\sum_{i=1}^{4} \lambda_i \leqslant 4$ from IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f_1 + f_2 = f$ derived in the previous section. However, the expressions of the CGCs for any n are too cumbersome to be tabulated. While the ISFs of $SO(n) \supset SO(n-1)$ for any n, which can be obtained according to Racah factorization lemma [4], are concise and easily to be listed in a table. For example, one can easily read off the following ISFs of $SO(n) \supset SO(n-1)$ with $n \geqslant 4$ from (46):

$$\left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [0] & [0] & [0] \end{array} \right\rangle = \sqrt{\frac{n-1}{n}} \qquad
\left\langle \begin{array}{cc|c} [1] & [1] & [2] \\ [1] & [1] & [0] \end{array} \right\rangle = -\sqrt{\frac{1}{n}}. \tag{53}$$

The notation for the ISFs used in (53) is much simpler than that of CGCs. Therefore, we shall only list ISFs of $SO(n) \supset SO(n-1)$ for $n \geqslant 4$ in the next section.

5. ISFs of $SO(n) \supset SO(n-1)$

In this section, we will list some ISFs of $SO(n) \supset SO(n-1)$ derived by using the assimilation method outlined in section 4. According to Racah factorization lemma, $SO(n)$

CGCs in the canonical basis $SO(n) \supset SO(n-1) \supset \cdots \supset SO(2)$ can be expressed as

$$\left(\begin{matrix} [\lambda_{1n}] & [\lambda_{2n}] \\ [\lambda_{1\ n-1}] & [\lambda_{2\ n-1}] \\ \cdots & \cdots \\ m_{12} & m_{22} \end{matrix}\left|\begin{matrix} \tau_n\ [\lambda_n] \\ [\lambda_{n-1}] \\ \cdots \\ m_2 \end{matrix}\right.\right) = \sum_{\tau_{n-1}} \left\langle \begin{matrix} [\lambda_{1n}] & [\lambda_{2n}] \\ [\lambda_{1\ n-1}] & [\lambda_{2\ n-1}] \end{matrix} \right| \left. \begin{matrix} \tau_n\ [\lambda_n] \\ \tau_{n-1}\ [\lambda_{n-1}] \end{matrix} \right\rangle \times \left(\begin{matrix} [\lambda_{1\ n-1}] & [\lambda_{2\ n}] \\ [\lambda_{1\ n-2}] & [\lambda_{2\ n-2}] \\ \cdots & \cdots \\ m_{12} & m_{22} \end{matrix}\left|\begin{matrix} \tau_{n-1}\ [\lambda_{n-1}] \\ [\lambda_{n-2}] \\ \cdots \\ m_2 \end{matrix}\right.\right) \tag{54}$$

where

$$\left(\begin{matrix} [\lambda_{1n}] & [\lambda_{2n}] \\ [\lambda_{1\ n-1}] & [\lambda_{2\ n-1}] \\ \cdots & \cdots \\ m_{12} & m_{22} \end{matrix}\left|\begin{matrix} \tau_n\ [\lambda_n] \\ [\lambda_{n-1}] \\ \cdots \\ m_2 \end{matrix}\right.\right) \tag{55}$$

is $SO(n)$ CGC,

$$\left\langle \begin{matrix} [\lambda_{1n}] & [\lambda_{2n}] \\ [\lambda_{1\ n-1}] & [\lambda_{2\ n-1}] \end{matrix} \right| \left. \begin{matrix} \tau_n\ [\lambda_n] \\ \tau_{n-1}\ [\lambda_{n-1}] \end{matrix} \right\rangle$$

is $SO(n) \supset SO(n-1)$ ISF, and τ_n is the multiplicity label needed in the coupling $[\lambda_{1n}] \times [\lambda_{2n}] \downarrow [\lambda_n]$. The ISFs satisfy the following orthogonality conditions

$$\begin{aligned} &\sum_{\lambda_{1\ n-1}\lambda_{2\ n-1}} \left\langle \begin{matrix} [\lambda_{1n}] & [\lambda_{2n}] \\ [\lambda_{1\ n-1}] & [\lambda_{2\ n-1}] \end{matrix} \right| \left. \begin{matrix} \tau_n\ [\lambda_n] \\ \tau_{n-1}\ [\lambda_{n-1}] \end{matrix} \right\rangle \left\langle \begin{matrix} [\lambda_{1n}] & [\lambda_{2n}] \\ [\lambda_{1\ n-1}] & [\lambda_{2\ n-1}] \end{matrix} \right| \left. \begin{matrix} \tau'_n\ [\lambda'_n] \\ \tau_{n-1}\ [\lambda_{n-1}] \end{matrix} \right\rangle \\ &\qquad = \delta_{\lambda'_n\lambda_n}\delta_{\tau'_n\tau_n} \\ &\sum_{\tau_n\lambda_n} \left\langle \begin{matrix} [\lambda_{1n}] & [\lambda_{2n}] \\ [\lambda_{1\ n-1}] & [\lambda_{2\ n-1}] \end{matrix} \right| \left. \begin{matrix} \tau_n\ [\lambda_n] \\ \tau_{n-1}\ [\lambda_{n-1}] \end{matrix} \right\rangle \left\langle \begin{matrix} [\lambda_{1n}] & [\lambda_{2n}] \\ [\lambda'_{1\ n-1}] & [\lambda'_{2\ n-1}] \end{matrix} \right| \left. \begin{matrix} \tau_n\ [\lambda_n] \\ \tau_{n-1}\ [\lambda_{n-1}] \end{matrix} \right\rangle \\ &\qquad = \delta_{\lambda'_{1\ n-1}\lambda_{1\ n-1}}\delta_{\lambda'_{2\ n-1}\lambda_{2\ n-1}}. \end{aligned} \tag{56}$$

In the following, we list modification-rule-free $SO(n) \supset SO(n-1)$ ISFs for the coupling $[\lambda_1] \times [\lambda_2]$ with resulting irreps $[\lambda_{1n}, \lambda_{2n}, \lambda_{3n}, \lambda_{4n}, \dot{0}]$ for $\sum_i \lambda_{in} \leqslant 4$, which are obtained from IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f_1 + f_2 = f \leqslant 4$.

6. Conclusions

In this paper, induced representations of $D_f(n)$ from $S_{f_1} \times S_{f_2}$ with $f_1 + f_2 = f$ are constructed. The IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ with $f \leqslant 4$ up to a normalization factor are derived by using the linear equation method. It is found that these IDCs are $SO(n)$ tensor component dependent. Weyl tableaux for the corresponding Gel'fand basis of $SO(n)$ are defined. The assimilation method for obtaining CGCs of $SO(n)$ in the Gel'fand basis with no modification rule involved couplings from IDCs of Brauer algebras are proposed, which is based on the Brauer–Schur–Weyl duality relation between $O(n)$ and Brauer algebra $D_f(n)$. ISFs of $SO(n) \supset SO(n-1)$ for the resulting irrep $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dot{0}]$ with $\sum_{i=1}^{4} \lambda_i \leqslant 4$ are tabulated. From these tables of ISFs, one can find that there are two types of modification-rule-free ISFs of $SO(n) \supset SO(n-1)$ or CGCs of $SO(n)$ in its canonical basis. The type-one ISFs or CGCs are n-independent, which are the same as ISFs of $U(n) \supset U(n-1)$ or CGCs of $U(n)$ in the canonical basis. Therefore, the following type of ISFs of $U(n) \supset U(n-1)$ are also $SO(n) \supset SO(n-1)$ ISFs:

$$\left\langle \begin{matrix} [\lambda_1] & [\lambda_2] \\ [\nu_1] & [\nu_2] \end{matrix} \middle| \begin{matrix} \tau_n[\lambda'] \\ \tau_{n-1}[\nu'] \end{matrix} \right\rangle \tag{57}$$

with $\sum_i \lambda'_i - \sum_j \nu'_j = 0$ or 1. Hence, ISFs for $U(n) \supset U(n-1)$ of this type or CGCs for $U(n)$ of this type derived previously [21–23], in which many results are with outer multiplicity, are also those of $SO(n)$. From a Brauer algebra point of view, there is no trace contraction between $[\lambda_1]$ and $[\lambda_2]$ in $SO(n)$ coupings, and in the reductions of $SO(n) \downarrow SO(n-1)$ in these cases. As a consequence, these coefficients can be derived from the IDCs of $S_{f_1} \times S_{f_2} \uparrow S_f$, which has already been discussed in [21]. The type-two ISFs of $SO(n) \supset SO(n-1)$ (CGCs of $SO(n)$ in the canonical basis) are rank-n dependent, which can only be obtained from IDCs of $S_{f_1} \times S_{f_2} \uparrow D_f(n)$. However, how to derive modification rule involved CGCs of $SO(n)$ from the IDCs of Brauer algebras still needs further study.

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